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Linear Poisson structures on $\mathbb{R}^4$ ☆

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Abstract

We classify all of the 4-dimensional linear Poisson structures of which the corresponding Lie algebras can be considered as the extension by a derivation of 3-dimensional unimodular Lie algebras. The affine Poisson structures on $\mathbb{R}^3$ are totally classified.

Keywords: Linear Poisson structure; Jacobi structure; Cohomology; Extension; Affine Poisson structure

1. Introduction

Linear Poisson structures are in one-to-one correspondence with Lie algebra structures and are usually called Lie–Poisson structures, see [12] for more details. It is the most basic and important Poisson structure both for its exquisite algebraic and geometric properties and for its far and wide applications in physics and other fields of mathematics. In [8], the authors have classified linear Poisson structures on $\mathbb{R}^3$, i.e., give the classification of Lie algebras on $\mathbb{R}^3$, see also [4] for more details which gives the classification of 3-dimensional Lie algebras on algebraic closed field. The idea of using linear Poisson structures to understand the structures of Lie algebras can be traced back to the work of Lie. In this spirit, there have been some suggestions for pursuing this geometric approach for Lie algebra structures (e.g., see [1,3]).

A natural problem is to classify linear Poisson structures on $\mathbb{R}^4$. We find that any 4-dimensional Lie algebra is the extension of some unimodular 3-dimensional Lie algebra by the viewpoint of Poisson geometry, so based on the results of the classification of 3-dimensional Lie algebras in [8] and after the computation of cohomology groups, linear Poisson structures on $\mathbb{R}^4$ are totally classified. Furthermore, as mentioned in [10], the affine Poisson structures are in one-to-one correspondence with central extension of the corresponding Lie algebras and the affine Poisson structures on $\mathbb{R}^3$ are totally classified. Finally, we give an example of Jacobi manifold of which the leaves enjoy conformal symplectic structure.

The paper is organized as follows: In Section 2 we briefly review the decomposition of linear Poisson structures on $\mathbb{R}^n$ that was done in [8] and concentrate on $\mathbb{R}^3$ and $\mathbb{R}^4$, obtain the result that any 4-dimensional Lie algebra is the extension of some unimodular 3-dimensional Lie algebra. In Section 3 we list some useful results that related with the classification of linear Poisson structures on $\mathbb{R}^3$ which will be used when we consider the classification of Lie–Poisson structures on $\mathbb{R}^4$. In Section 4 we give a detail description of cohomology groups with coefficients in

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trivial representation and adjoint representation of 3-dimensional Lie algebras. In the last section we first consider all possible extensions of 3-dimensional Lie algebras and then linear Poisson structures on $\mathbb{R}^4$ are totally classified. We obtain the result of the classification of affine Poisson structures $\mathbb{R}^3$.

2. The decomposition of Lie–Poisson structures

In this section, we first review the decomposition of linear Poisson structures on $\mathbb{R}^n$ [8], motivation given in [9] and based on this, we mainly concentrate on Lie–Poisson structures on $\mathbb{R}^3$ and $\mathbb{R}^4$. Consider Lie algebras corresponding to them, we find that every 4-dimensional Lie algebra is the extension, central extension or extension by a derivation, of some unimodular 3-dimensional Lie algebra.

Throughout the paper, $\mathfrak{g}$ will be a n-dimensional Lie algebra and $\mathfrak{g}^*$ its dual space with Lie–Poisson structure $\pi_{\mathfrak{g}}$ on it. $e_1, \dots, e_n$ are the basis of Lie algebra $\mathfrak{g}$, the corresponding coordinate functions are $x_1, \dots, x_n$ and $e^1, \dots, e^n$ are the dual basis of $\mathfrak{g}^*$, the corresponding coordinate functions are $x^1, \dots, x^n$. In [8], the authors have given the decomposition of Lie–Poisson structures on $\mathbb{R}^n$, let us recall them briefly.

Let $\Omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \cdots \wedge \mathrm{d}x_n$ be the canonical volume form on $\mathbb{R}^n$. Then Ω induces an isomorphism Φ from the space of all i-multiple vector fields to the space of all $(n-i)$-forms. Let d denote the usual exterior differential on forms and

$$D = (-1)^{k+1}\Phi^{-1} \circ \mathrm{d} \circ \Phi : \mathcal{X}^k(\mathbb{R}^n) \rightarrow \mathcal{X}^{k-1}(\mathbb{R}^n),$$

its pull-back under the isomorphism Φ, where $\mathcal{X}^k(\mathbb{R}^n)$ denotes the space of all k-multiple vector fields on $\mathbb{R}^n$. An important property of D is that the Schouten bracket can be written in terms of this operator as follows [6]:

$$[U, V] = D(U \wedge V) - D(U) \wedge V - (-1)^i U \wedge D(V), \tag{1}$$

for all $U \in \mathcal{X}^i(\mathbb{R}^n)$ and $V \in \mathcal{X}^j(\mathbb{R}^n)$. It is obvious that there is a one-to-one correspondence between matrices in $\mathfrak{gl}(\mathfrak{n})$ and linear vector fields on $\mathbb{R}^n$, i.e.,

$$A = (a_{ij}) \longleftrightarrow \hat{A} = \sum_{ij} a_{ij} x_j \frac{\partial}{\partial x_i}, \qquad \mathrm{div}_\Omega\, \hat{A} = D(\hat{A}) = \mathrm{tr}\, A. \tag{2}$$

Moreover, a vector $k \in \mathbb{R}^n$ corresponds to a constant vector field $\hat{k}$ by translation on $\mathbb{R}^n$ and satisfies

$$\mathrm{div}_\Omega\, \hat{k} = D(\hat{k}) = 0, \qquad [\hat{A}, \hat{k}] = -\widehat{Ak}, \quad \forall A \in \mathfrak{gl}(\mathfrak{n}). \tag{3}$$

For a given Poisson tensor π, let $D(\pi)$ be its modular vector field (see [11], which is also called the curl vector field in [2]). Such a vector field is always compatible with π, i.e.,

$$L_{D(\pi)}\pi = [D(\pi), \pi] = 0. \tag{4}$$

A Poisson structure is called unimodular if $D(\pi) = 0$. For a linear Poisson structure π on $\mathbb{R}^n$, there exists some $k \in \mathbb{R}^n$ such that $D(\pi) = \widehat{k}$ and k is also called the modular vector of π. In fact k is always invariant if one takes a different volume form. As mentioned in [11], k is the modular character of Lie algebra $\mathfrak{g}$ which corresponds to the linear Poisson structure defined as a vector in $\mathfrak{g}^*$ such that

$$\langle k, \xi \rangle = \mathrm{tr} \circ \mathrm{ad}(\xi), \quad \forall \xi \in \mathfrak{g}.$$

Theorem 2.1 (*[8]*). *Any Lie–Poisson structure $\pi_{\mathfrak{g}}$ on $\mathfrak{g}^* \cong \mathbb{R}^n$ has a unique decomposition:*

$$\pi_{\mathfrak{g}} = \frac{1}{n-1}\hat{I} \wedge \hat{k} + \Lambda_{\mathfrak{g}}, \tag{5}$$

where $k \in \mathbb{R}^n$ is the modular vector of $\pi_{\mathfrak{g}}$ and $\Lambda_{\mathfrak{g}}$ is a linear bi-vector field satisfying $D(\Lambda_{\mathfrak{g}}) = 0$ and $(\frac{1}{n-1}\hat{k}, \Lambda_{\mathfrak{g}})$ is a Jacobi structure.

Conversely, any such pair satisfying the above compatibility conditions defines a Lie–Poisson structure by Formula (5).

Remark 2.2. For more information about Jacobi structures, please see [5,7].

Next we give some language of cohomology groups and extension of a Lie algebra which will be often used later and then based on Theorem 2.1, we obtain the first result that every 4-dimensional Lie algebra is the extension of some unimodular 3-dimensional Lie algebra.

Recall that for any Lie algebra $\mathfrak{g}$ and its representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ on a vector space V, we have the standard *Chevalley cochain complex* $C^k = \mathrm{Hom}(\wedge^k \mathfrak{g}, V)$ and the coboundary operator $\delta^k : C^k \longrightarrow C^{k+1}$ is given by

$$(\delta f)(\xi_1, \cdots, \xi_{k+1}) = \sum_{i=1}^{k+1}(-1)^i (\rho\xi) f(\xi_1, \ldots, \widehat{\xi_i}, \ldots, \xi_{k+1}) + \sum_{i<j}(-1)^{i+j} f([\xi_i, \xi_j], \xi_1, \ldots, \widehat{\xi_i}, \ldots, \widehat{\xi_j}, \ldots, \xi_{k+1}), \quad \forall f \in C^k, \xi_1, \ldots, \xi_{k+1} \in \mathfrak{g}.$$

In particular, $\delta^0 : C^0 = V \longrightarrow C^1$ is given by

$$\delta^0 v(\xi) = (\rho\xi)(v), \quad \forall v \in V, \xi \in \mathfrak{g}. \tag{6}$$

There are two natural representations of $\mathfrak{g}$ which are trivial representation on $\mathbb{R}$ and adjoint representation on itself.

The expression "$\mathfrak{h}$ is the *central extension* of $\mathfrak{g}$ by $\mathbb{R}$" means that one has a well defined exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R} \xrightarrow{\ \iota\ } \mathfrak{h} \xrightarrow{\ \kappa\ } \mathfrak{g} \longrightarrow 0,$$

where $\iota(\mathbb{R})$ belongs to the center of $\mathfrak{h}$. This shows that $\mathfrak{h} = \mathfrak{g} \oplus \mathbb{R}$ as a vector space, and we must have

$$[\xi \oplus t, \eta \oplus s]_{\mathfrak{h}} = [\xi, \eta] \oplus \omega(\xi, \eta),$$

where ω is a *Chevalley–Eilenberg* 2-cocycle of $\mathfrak{g}$. Furthermore, there is also a one-to-one correspondence between *affine Poisson structures (or modified Lie–Poisson structures)* on $\mathfrak{g}^*$ and central extensions of $\mathfrak{g}$. See more details in [10]. For convenience, denote by $\mathfrak{g}_\omega$ the extension of $\mathfrak{g}$ decided by ω.

Given a 3-dimensional Lie algebra $\mathfrak{g}$ with bracket $[\cdot, \cdot]$ and a derivation D, we can define a new 4-dimensional Lie algebra as the extension of $\mathfrak{g}$ by the derivation D (*D-extension* for convenience), denoted by $\mathfrak{g}_D$, $\mathfrak{g}_D = \mathfrak{g} \oplus \mathbb{R}e$, with bracket $[\cdot, e]_D = D(\cdot)$.

It is known that if $\omega_1 - \omega_2$ is exact, $\mathfrak{g}_{\omega_1}$ is isomorphic to $\mathfrak{g}_{\omega_2}$. If $D_1 - D_2$ is exact, $\mathfrak{g}_{D_1}$ is isomorphic to $\mathfrak{g}_{D_2}$.

Proposition 2.3. *Any* 4-*dimensional Lie algebra is the extension, central extension or D-extension, of some unimodular* 3-*dimensional Lie algebra.*

Proof. As stated in the Theorem 2.1, the Lie–Poisson structure on $\mathfrak{g}^*$ has the decomposition $\pi_{\mathfrak{g}} = \frac{1}{3}\widehat{I} \wedge \widehat{k} + \Lambda_{\mathfrak{g}}$. First we consider the case that $k = 0$.

$\pi_{\mathfrak{g}} = \Lambda_{\mathfrak{g}}$ is the Lie–Poisson structure on $\mathfrak{g}^*$, so $[\Lambda_{\mathfrak{g}}, \Lambda_{\mathfrak{g}}] = 0$. Using Formula (1), we can easily get that $D(\Lambda_{\mathfrak{g}} \wedge \Lambda_{\mathfrak{g}}) = 0$ and $\Lambda_{\mathfrak{g}} \wedge \Lambda_{\mathfrak{g}} = 0$.

Assume that

$$\Lambda_{\mathfrak{g}} = \sum_{i,j,k=1}^{4} C_{ij}^k x^k \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} = \sum_{l=1}^{4} x^l \pi_l,$$

where $\pi_l = \sum_{i,j} C_{ij}^l \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$. We have

$$\Lambda_{\mathfrak{g}} \wedge \Lambda_{\mathfrak{g}} = 0 \Longleftrightarrow x^l \pi_l \wedge x^l \pi_l = 0 \Longleftrightarrow \pi_l \wedge \pi_h = 0,$$

for any $l, h = 1, 2, 3, 4$. So π_l decides a 2-dimensional subspace P_l and P_l, P_h intersect a line. If there exists some $\pi_l = 0$ or some P_l which is the linear composition of the others, the problem is easier and we can omit them.

Then consider the lines which are the intersection of some two subspaces P_l and P_h, there may be three cases below:

(1) There are two different subspaces at least and all the subspaces intersect only one line L;
(2) There are three linear independent lines L_1, L_2, L_3;
(3) All P_l coincide.

As in the Case (1), let $L = \mathbb{R}e^4 = \cap P_l \in \mathfrak{g}^*$ and $\pi = X \wedge \frac{\partial}{\partial x^4}$ for some linear vector field X on $\mathfrak{g}^*$. Denote $L^0 = \ker(e^4) \subset \mathfrak{g}$ and so $\mathfrak{g} = L^0 \oplus \mathbb{R}e_4$. We will show that L^0 is a 3-dimensional Abelian idea, and then follows that Lie algebra $\mathfrak{g}$ is the D-extension of 3-dimensional Abelian Lie algebra L^0. For any $\xi, \eta \in L^0$,

$$[\xi, \eta] = \pi(\xi, \eta) = X \wedge \frac{\partial}{\partial x^4}(\xi, \eta) = 0,$$

since $\langle \frac{\partial}{\partial x^4}, \xi\rangle = \langle \frac{\partial}{\partial x^4}, \eta\rangle = 0$. Furthermore we have $\langle e^4, [\xi, e_4]\rangle = 0$ for any $\xi \in \mathfrak{g}$ since $D(\pi) = 0$ and this implies $[\xi, e_4] \in L^0$ and follows that L^0 is the Abelian idea of Lie algebra $\mathfrak{g}$.

As in the Case (2), the three linear independent lines L_1, L_2, L_3 expand a 3-dimensional subspace H and $H^0 = \ker(H) \in \mathfrak{g}$ is a 1-dimensional subspace of $\mathfrak{g}$. Let $H^0 = \mathbb{R}e_4$ and choose a 3-dimensional subspace E of $\mathfrak{g}$ such that $\mathfrak{g} = H^0 \oplus E$. We will show that e_4 is the center of Lie algebra $\mathfrak{g}$ and follows that $\mathfrak{g}$ is the central extension of Lie algebra $E(\text{mod } e_4)$. Let $L_1 = \mathbb{R}e^1$, $L_2 = \mathbb{R}e^2$, $L_3 = \mathbb{R}e^3$, so

$$\pi = \sum_{k=1}^{4}\left(C_{12}^k x^k \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + c_{13}^k x^k \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + C_{23}^k x^k \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}\right),$$

for some constants $C_{12}^k, C_{13}^k, C_{23}^k$. Obviously for any $\xi \in E$,

$$[\xi, e_4] = \pi(\xi, e_4) = 0,$$

so e_4 is the center of Lie algebra $\mathfrak{g}$.

As in the Case (3), choose a line $L \in P_l$ and with the same method as in the Case (1), we can get the conclusion that it is the D-extension of some 3-dimensional Abelian Lie algebra. In fact, it is a particular case of Case (1).

When $D(\pi) = k \neq 0$, k is the modular character of Lie algebra $\mathfrak{g}$ defined as a vector in $\mathfrak{g}^*$ such that

$$\langle k, \xi\rangle = \mathrm{tr}(\mathrm{ad}(\xi)), \quad \forall \xi \in \mathfrak{g}.$$

Let $\mathfrak{h} = \ker k$, $\forall \xi \in \mathfrak{h}$, $\mathrm{tr}(\mathrm{ad}(\xi)) = 0$. $\mathfrak{h}^* = (\ker k)^* = \mathfrak{g}^*/\mathbb{R}k \subset \mathfrak{g}^*$, the Lie–Poisson structure on $\mathfrak{h}^*$ is just the reduction of $\Lambda_{\mathfrak{g}}$ on $\mathfrak{h}^*$. $\forall \eta \in \mathfrak{g}$,

$$\mathrm{tr}(\mathrm{ad}([\xi, \eta])) = \mathrm{tr}([\mathrm{ad}(\xi), \mathrm{ad}(\eta)]) = 0.$$

So $\mathfrak{h}$ is the idea of the Lie algebra $\mathfrak{g}$, and Lie algebra $\mathfrak{g}$ is *D-extension* of Lie algebra $\mathfrak{h}$.

It is evident that the Abelian Lie algebra and Lie algebra $E(\text{mod } e_4)$ is unimodular. The Lie–Poisson structure associated with $\mathfrak{h}$ is just the reduction of $\Lambda_{\mathfrak{g}}$ on $\mathfrak{h}^*$, so $\mathfrak{h}$ is unimodular. This completes the proof of the proposition. ■

Remark 2.4. If the modular character k of the 4-dimension Lie algebra is $(0, 0, 0, 1)^T \in \mathbb{R}^4$, we should consider the D-extension of which the trace of the derivation D is not zero, since the modular character of $\mathfrak{g}_D$ is $(0, 0, 0, \mathrm{tr}(D))^T$.

So before we study the classification of linear Poisson structures on $\mathbb{R}^4$, we should first study the classification of linear Poisson structures on $\mathbb{R}^3$ which is done in [8] and the cohomology groups of 3-dimensional Lie algebras with coefficients in trivial representation and adjoint representation. This is shown the following two sections.

3. The classification of Lie–Poisson structures on $\mathbb{R}^3$

In this section, we list some useful results in [8] about the classification of Lie–Poisson structures on $\mathbb{R}^3$.

In [8], it is pointed out that Lie–Poisson structures $\pi_{\mathfrak{g}}$ on $\mathbb{R}^3$ are in one-to-one correspondence with *compatible pair* (k, f) and denoted by $\pi_{k,f}$, where k is the modular vector and f is a quadratic function, such that $\hat{k} f = 0$ and

$$\pi_{\mathfrak{g}} = \pi_{k,f} = \frac{1}{2}\widehat{I} \wedge \widehat{k} + \pi_f = \frac{1}{2}\widehat{I} \wedge \widehat{k} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}. \tag{7}$$

Theorem 3.1 (*[8]*). *Let π_1 and π_2 be two linear Poisson structures on $\mathbb{R}^3$ determined by the compatible pairs (k_1, f_1) and (k_2, f_2) respectively. Then π_1 is isomorphic to π_2 if and only if there is a $T \in GL(3)$ such that*

$$k_2 = Tk_1, \qquad f_2 = \det(T) f_1 \circ T^{-1}.$$

Corollary 3.2. *With notations given above, consider the automorphism group of* $\pi_{\mathfrak{g}}$ *and derivation of* $\mathfrak{g}$*, one has*

$$\mathrm{Aut}(\pi_{\mathfrak{g}}) = \{T \mid T \in GL(3), Tk = k, f \circ T = \det(T) f\}, \tag{8}$$

and

$$\mathrm{Der}(\mathfrak{g}) \cong \{D \mid D \in \mathfrak{g}l(3), Dk = 0, \hat{D} f = (\mathrm{tr}\, D) f\}. \tag{9}$$

Theorem 3.3 (*[8]*). *Any Lie–Poisson structure* $\pi_{k,f}$ *on* $\mathbb{R}^3$ *is isomorphic to one of the following standard forms:*

(A) $k = 0$ *(unimodular case),*	(B) $k = (0, 0, 1)^T$, *i.e.,* $\hat{k} = \frac{\partial}{\partial z}$,
(1) $f = 0$,	(7) $f = 0$,
(2) $f = x^2 + y^2 + z^2$,	(8) $f = a(x^2 + y^2)$,
(3) $f = x^2 + y^2 - z^2$,	(9) $f = a(x^2 - y^2)$,
(4) $f = x^2 + y^2$,	(10) $f = x^2$,
(5) $f = x^2 - y^2$,	
(6) $f = x^2$,	

where $a > 0$ *is a constant.*

Theorem 3.4 (*[8]*). *Let* $G_i, i = 1, \ldots, 10$ *denote the automorphism groups of the Lie–Poisson structures which corresponds to Case* (1) *in Theorem* 3.3*. Then we have*

$$G_1 = GL(3),$$
$$G_2 = SO(3),$$
$$G_3 = SO(2, 1),$$
$$G_4 = \left\{ \begin{pmatrix} \lambda T & 0 \\ \xi & \det T \end{pmatrix} \mid T \in O(2), \lambda \neq 0, \xi \in \mathbb{R}^2 \right\},$$
$$G_5 = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \alpha & 0 \\ \gamma & \delta & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -\alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ \gamma & \delta & -1 \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}, \alpha^2 \neq \beta^2 \right\},$$
$$G_6 = \left\{ \begin{pmatrix} a & 0 \\ \xi & A \end{pmatrix} \mid A \in GL(2), \det(A) = a \neq 0, \xi \in \mathbb{R}^2 \right\},$$
$$G_7 = \left\{ \begin{pmatrix} A & 0 \\ \xi & 1 \end{pmatrix} \mid A \in GL(2), \xi \in \mathbb{R}^2 \right\},$$
$$G_8 = \left\{ \begin{pmatrix} \lambda T & 0 \\ \xi & 1 \end{pmatrix} \mid T \in SO(2), \lambda \neq 0, \xi \in \mathbb{R}^2 \right\},$$
$$G_9 = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \alpha & 0 \\ \gamma & \delta & 1 \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}, \alpha^2 \neq \beta^2 \right\},$$
$$G_{10} = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \alpha & 0 \\ \gamma & \delta & 1 \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}, \alpha \neq 0 \right\}.$$

4. The cohomology groups of 3-dimensional Lie algebras

In this section, $\mathfrak{g}$ will be a 3-dimensional Lie algebra with modular character k and $\mathfrak{g}^*$ is its dual space with Lie–Poisson structure $\pi_{k,f}$, described as in (7), where f is decided by symmetric matrix A.

Theorem 4.1. *For any $\eta \in \mathfrak{g}^*$, $\omega \in \mathfrak{g}^* \wedge \mathfrak{g}^*$, δ is the coboundary operator of Lie algebra cohomology with coefficients in its trivial representation, then*

$$\delta\eta = \frac{1}{2}k \wedge \eta - 2\Phi \circ A\eta, \tag{10}$$

$$\delta\omega = k \wedge \omega. \tag{11}$$

Proof. By Theorem 2.1, $\pi = \frac{1}{2}\hat{I} \wedge \hat{k} + \Lambda_{\mathfrak{g}}$, and

$$\Lambda_{\mathfrak{g}} = \pi_f = \frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + \frac{\partial f}{\partial x_3}\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}.$$

So we have

$$[e_1, e_2] = \frac{1}{2}(\langle k, e_2\rangle e_1 - \langle k, e_1\rangle e_2) + 2Ae^3,$$
$$[e_2, e_3] = \frac{1}{2}(\langle k, e_3\rangle e_2 - \langle k, e_2\rangle e_3) + 2Ae^1,$$
$$[e_3, e_1] = \frac{1}{2}(\langle k, e_1\rangle e_3 - \langle k, e_3\rangle e_1) + 2Ae^2,$$

and

$$\begin{aligned}\langle \delta\eta, e_1 \wedge e_2\rangle &= -\langle \eta, [e_1, e_2]\rangle \\ &= -\left\langle \eta, \frac{1}{2}(\langle k, e_2\rangle e_1 - \langle k, e_1\rangle e_2) + 2Ae^3\right\rangle \\ &= \frac{1}{2}\langle k, e_1\rangle\langle \eta, e_2\rangle - \frac{1}{2}\langle k, e_2\rangle\langle \eta, e_1\rangle - 2\langle A\eta, \Phi(e_1 \wedge e_2)\rangle \\ &= \left\langle \frac{1}{2}k \wedge \eta, e_1 \wedge e_2\right\rangle - 2\langle \Phi \circ A\eta, e_1 \wedge e_2\rangle.\end{aligned}$$

After the similar computation of $\langle \delta\eta, e_2 \wedge e_3\rangle$ and $\langle \delta\eta, e_3 \wedge e_1\rangle$, we have the conclusion that

$$\delta\eta = \frac{1}{2}k \wedge \eta - 2\Phi \circ A\eta.$$

For $\eta_1, \eta_2 \in \mathfrak{g}^*$, by Leibnitz rule, we have

$$\begin{aligned}\delta(\eta_1 \wedge \eta_2) &= \delta\eta_1 \wedge \eta_2 - \eta_1 \wedge \delta\eta_2 \\ &= \left(\frac{1}{2}k \wedge \eta_1 - 2\Phi \circ A\eta_1\right) \wedge \eta_2 - \eta_1 \wedge \left(\frac{1}{2}k \wedge \eta_2 - 2\Phi \circ A\eta_2\right) \\ &= k \wedge \eta_1 \wedge \eta_2 + 2(\eta_1 \wedge (\Phi \circ A\eta_2) - (\Phi \circ A\eta_1) \wedge \eta_2) \\ &= k \wedge \eta_1 \wedge \eta_2.\end{aligned}$$

Thus for any $\omega \in \mathfrak{g}^* \wedge \mathfrak{g}^*$, $\delta\omega = k \wedge \omega$. ■

For Lie algebras $\mathfrak{g}$ listed in Theorem 3.3, we give a detail description of the corresponding 2-cocycles, denoted by $\mathcal{C}^2(\mathfrak{g}) \subset \mathfrak{g}^* \wedge \mathfrak{g}^*$ and exact 2-cocycles, denoted by $\mathcal{B}^2(\mathfrak{g}) \subset \mathfrak{g}^* \wedge \mathfrak{g}^*$, which will be used when considering the classification of linear Poisson structures on $\mathbb{R}^4$. Furthermore we give cohomology groups $H^1(\mathfrak{g})$, $H^2(\mathfrak{g})$, $H^3(\mathfrak{g})$ with coefficients in trivial representation.

Corollary 4.2. (A) $k = 0$ (*unimodular case*)

Quadratic function f	$\mathcal{C}^2(\mathfrak{g})$	$\mathcal{B}^2(\mathfrak{g})$	$H^1(\mathfrak{g})$	$H^2(\mathfrak{g})$	$H^3(\mathfrak{g})$
(1) 0	$\forall\omega$	0	$\mathbb{R}^3$	$\mathbb{R}^3$	$\mathbb{R}$
(2) $x^2 + y^2 + z^2$	$\forall\omega$	$\forall\omega$	0	0	$\mathbb{R}$
(3) $x^2 + y^2 - z^2$	$\forall\omega$	$\forall\omega$	0	0	$\mathbb{R}$

Quadratic function f	$\mathcal{C}^2(\mathfrak{g})$	$\mathcal{B}^2(\mathfrak{g})$	$H^1(\mathfrak{g})$	$H^2(\mathfrak{g})$	$H^3(\mathfrak{g})$
(4) x^2+y^2	$\forall\omega$	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
(5) x^2-y^2	$\forall\omega$	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
(6) x^2	$\forall\omega$	$\alpha dy\wedge dz$	$\mathbb{R}^2$	$\mathbb{R}^2$	$\mathbb{R}$

(B) $k=(0,0,1)^T$, *i.e.*, $\hat{k}=\frac{\partial}{\partial z}$

Quadratic function f	$\mathcal{C}^2(\mathfrak{g})$	$\mathcal{B}^2(\mathfrak{g})$	$H^1(\mathfrak{g})$	$H^2(\mathfrak{g})$	$H^3(\mathfrak{g})$
(7) 0	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\mathbb{R}$	0	0
(8) $a(x^2+y^2)$	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\mathbb{R}$	0	0
$(9)_1$ $a(x^2-y^2), a\neq\frac{1}{4}$	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\mathbb{R}$	0	0
$(9)_2$ $\frac{1}{4}(x^2-y^2)$	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\alpha(dy\wedge dz-dz\wedge dx)$	$\mathbb{R}^2$	$\mathbb{R}$	0
(10) x^2	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\alpha dy\wedge dz+\beta dz\wedge dx$	$\mathbb{R}$	0	0

where α,β are arbitrary constants.

Proof. First consider H^3, by Theorem 4.1, for $\forall\omega\in\mathfrak{g}^*\wedge\mathfrak{g}^*$, $\delta\omega=k\wedge\omega$. So in the unimodular case $k=0$, this implies that $\delta\omega=0$. So $\theta\in\wedge^3\mathfrak{g}^*$ is exact if and only if $\theta=0$, and this implies that $H^3=\mathbb{R}$. If $k=(0,0,1)$, $\forall\,\theta\in\wedge^3\mathfrak{g}^*$ is exact, so $H^3=0$.

Next consider H^1, there is no exact chain and $\eta\in\mathfrak{g}^*$ is closed if and only if

$$\frac{1}{2}k\wedge\eta-2\Phi\circ A\eta=0$$

by Theorem 4.1. If $k=0$, $A\eta=0$, so the dimension of the first cohomology group is 3 − order(A) and we have the conclusions listed above. If $k=(0,0,1)$, it is a little complicated but straightforward however and we leave it to the interest of the reader.

Finally we consider H^2, in the case that $k=0$, all of 2-chains are 2-cocycles by Theorem 4.1. In the case that $k=(0,0,1)$, $\omega\in\wedge^2\mathfrak{g}^*$ is closed if and only if ω has the form $\omega=\alpha dy\wedge dz+\beta dz\wedge dx$ by Theorem 4.1, where $\alpha,\beta\in\mathbb{R}$. $\omega\in\wedge^2\mathfrak{g}^*$ is exact if and only if

$$\omega=\frac{1}{2}k\wedge\eta-2\Phi\circ A\eta$$

for some $\eta\in\mathfrak{g}^*$ by Theorem 4.1, the conclusion is straightforward. ■

Remark 4.3. In fact, the first cohomology group $H^1(\mathfrak{g})$ relate to the dimension of derived algebra $[\mathfrak{g},\mathfrak{g}]$ of 3-dimensional Lie algebra $\mathfrak{g}$, more precisely, $\dim(H^1(\mathfrak{g}))=3-\dim([\mathfrak{g},\mathfrak{g}])$. In the unimodular case, $\dim([\mathfrak{g},\mathfrak{g}])=\mathrm{order}(A)$, however it is not true if the modular vector is not zero (see the case $f=\frac{1}{4}(x^2-y^2)$).

The next theorem and corollary give some description of the derivation which will be used in the next section when we consider the extension of a Lie algebra by a derivation.

Theorem 4.4. *Let $\mathfrak{g}$ be one of the Lie algebras listed in Theorem 3.3. D is a derivation of Lie algebra $\mathfrak{g}$ if and only if*

$$D^*k=0,\qquad DA+AD^*=\mathrm{tr}(D)A. \tag{12}$$

And D is an inner derivation if and only if there exists a skew-symmetric transformation $B:\mathfrak{g}\to\mathfrak{g}^*$ such that

$$D=\left(2A+\frac{1}{2}\overline{k}\right)B, \tag{13}$$

where $\overline{k}:\mathfrak{g}^*\longrightarrow\mathfrak{g}$ denotes the skew-symmetric transformation decided by $\Phi^{-1}k$.

Proof. Since D is a derivation, by Corollary 3.2 we have

$$D^*k = 0, \qquad \hat{D}^* f = \mathrm{tr}(D) f,$$

and this implies that

$$D^*k = 0, \qquad DA + AD^* = \mathrm{tr}(D)A.$$

If D is an inner derivation, there exists a $\xi \in \mathfrak{g}$, such that $D = \mathrm{ad}(\xi)$ and $DX = \mathrm{ad}(\xi)(X) = [\xi, X]$. Furthermore,

$$\begin{aligned}[\xi, X] &= c.p.\{(\langle \xi, e^1\rangle\langle X, e^2\rangle - \langle \xi, e^2\rangle\langle X, e^1\rangle)[e_1, e_2]\}\\ &= c.p.\left\{(\langle \xi, e^1\rangle\langle X, e^2\rangle - \langle \xi, e^2\rangle\langle X, e^1\rangle)\left(\frac{1}{2}(\langle k, e_2\rangle e_1 - \langle k, e_1\rangle e_2) + 2Ae^3\right)\right\}\\ &= c.p.\left\{\frac{1}{2}\Phi^{-1}(k \wedge (\langle \xi, e^1\rangle\langle X, e^2\rangle - \langle \xi, e^2\rangle\langle X, e^1\rangle)e^3)\right\} + 2A\Phi(\xi \wedge X)\\ &= \frac{1}{2}\Phi^{-1}(k \wedge (\Phi(\xi \wedge X))) + 2A\Phi(\xi \wedge X),\end{aligned}$$

where $c.p.\{\}$ means the cyclic permutations of e^1, e^2, e^3. So we have

$$\begin{aligned}D(\cdot) &= \frac{1}{2}\Phi^{-1}(k \wedge (\Phi(\xi)(\cdot))) + 2A\Phi(\xi)(\cdot)\\ &= \left(2A + \frac{1}{2}\overline{k}\right) B(\cdot),\end{aligned}$$

where $\overline{B} : \mathfrak{g} \to \mathfrak{g}^*$ is the skew-symmetric transformation decided by $\Phi\xi$ and $\overline{k} : \mathfrak{g}^* \to \mathfrak{g}$ is the skew-symmetric transformation decided by $\Phi^{-1}k$. ■

Now we can give the form of the derivation and the inner derivation via the standard form of Lie algebras listed in Theorem 3.3. However inner derivation will be of no use when we consider the extension of a Lie algebra by a derivation.

Corollary 4.5. (A) $k = 0$ (*unimodular case*)

Quadratic function f	*Derivation*	*Inner derivation*	$H^1(\mathrm{ad}; \mathfrak{g})$
(1) 0	$\forall D \in \mathfrak{gl}(3)$	0	$\mathbb{R}^9$
(2) $x^2 + y^2 + z^2$	$D \in \mathfrak{o}(3)$	$D \in \mathfrak{o}(3)$	0
(3) $x^2 + y^2 - z^2$	$D \in \mathfrak{o}(2, 1)$	$D \in \mathfrak{o}(2, 1)$	0
(4) $x^2 + y^2$	$\begin{pmatrix} \alpha & \beta & \gamma \\ -\beta & \alpha & \delta \\ 0 & 0 & 0 \end{pmatrix}$	$\alpha = 0$	$\mathbb{R}$
(5) $x^2 - y^2$	$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \alpha & \delta \\ 0 & 0 & 0 \end{pmatrix}$	$\alpha = 0$	$\mathbb{R}$
(6) x^2	$\begin{pmatrix} \mathrm{tr}(\overline{D}) & \xi \\ 0 & \overline{D} \end{pmatrix}$, $\xi \in \mathbb{R}^2$, $\overline{D} \in \mathfrak{gl}(2)$	$\overline{D} = 0$	$\mathbb{R}^4$

(B) $k = (0, 0, 1)^T$, *i.e.*, $\hat{k} = \frac{\partial}{\partial z}$

Quadratic function f	*Derivation*	*Inner derivation*	$H^1(\mathrm{ad};\mathfrak{g})$
(7) 0	$\begin{pmatrix}\alpha & \beta & \gamma\\ \delta & \epsilon & \varepsilon\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}\epsilon & o & \gamma\\ 0 & \epsilon & \varepsilon\\ 0 & 0 & 0\end{pmatrix}$	$\mathbb{R}^3$
(8) $a(x^2+y^2)$	$\begin{pmatrix}\alpha & \beta & \gamma\\ -\beta & \alpha & \delta\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}\alpha & 4a\alpha & \gamma\\ -4a\alpha & \alpha & \delta\\ 0 & 0 & 0\end{pmatrix}$	$\mathbb{R}$
$(9)_1$ $a(x^2-y^2), a \neq \frac{1}{4}$	$\begin{pmatrix}\alpha & \beta & \gamma\\ \beta & \alpha & \delta\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}\alpha & 4a\alpha & \gamma\\ 4a\alpha & \alpha & \delta\\ 0 & 0 & 0\end{pmatrix}$	$\mathbb{R}$
$(9)_2$ $\frac{1}{4}(x^2-y^2)$,	$\begin{pmatrix}\alpha & \beta & \gamma\\ \beta & \alpha & \delta\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}\alpha & \alpha & \gamma\\ \alpha & \alpha & \gamma\\ 0 & 0 & 0\end{pmatrix}$	$\mathbb{R}^2$
(10) x^2	$\begin{pmatrix}\alpha & \beta & \gamma\\ 0 & \alpha & \delta\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}\alpha & 4\alpha & \gamma\\ 0 & \alpha & \delta\\ 0 & 0 & 0\end{pmatrix}$	$\mathbb{R}$

where $a > 0, \alpha, \beta, \gamma, \delta, \epsilon, \varepsilon \in \mathbb{R}$ *are arbitrary constants.*

Proof. The proof is almost straightforward and we only give the proof of the Case (2). Any derivation D satisfies $DA + AD^* = \mathrm{tr}(D^*)A$. Multiply A^{-1} at the right-hand side, we have $D = -AD^*A^{-1} + \mathrm{tr}(D^*)I$, and this implies that

$$\mathrm{tr}(D) = -\mathrm{tr}(D) + 3\mathrm{tr}(D) \Longrightarrow \mathrm{tr}(D) = 0.$$

Then $D = -AD^*A^{-1}$, this implies that D is skew-symmetric. To prove that D is an inner derivation, we only need to say that $-D^*A^{-1}$ is skew-symmetric. Notice that $DA + AD^* = 0$ means $D^*A^{-1} + A^{-1}D = 0$, this is just the condition that $-D^*A^{-1}$ is skew-symmetric. ■

5. The classification of linear Poisson structures on $\mathbb{R}^4$

With the above preparations, we can see that the procedure of classification of Lie–Poisson structures on $\mathbb{R}^4$ can be split into four steps by Theorem 3.3, Corollaries 4.2 and 4.5 as follows.

(1) Take a standard form of Lie–Poisson structure $\pi_{\mathfrak{g}}$ from the list in Theorem 3.3(A).
(2) Consider all the isomorphism classes of the extension, central extension and D-extension of the corresponding Lie algebra.
(3) Consider the isomorphism of the extension of two different Lie algebras.
(4) The Lie–Poisson structures corresponding to the above Lie algebras give the classification of linear Poisson structures on $\mathbb{R}^4$.

In fact Step (2) above is the most difficult and complicated one. If this is done, Step (3) is almost straightforward however. The following proposition gives a detail description of central extensions and D-extensions of Lie algebras listed in Theorem 3.3 and of which the first part will be used in the classification of 4-dimensional Lie–Poisson structures and the whole proposition will be of great importance when we consider *affine Poisson structures*.

Proposition 5.1. *With the same notations given above, let* $\mathfrak{g}$ *be one of the* 3*-dimensional Lie algebras listed in Theorem* 3.3*, any of its extension, central extension which is decided by* 2*-cocycles* ω *in trivial representation and D-extension which is decided by derivations, is isomorphic to one of the following forms:*

(A) $k = 0$ (*unimodular case*)

Quadratic function f	*2-cocycles* ω	*Derivation D*
(1) 0	0 *and* $\mathrm{d}x \wedge \mathrm{d}y$	$0, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
(2) $x^2 + y^2 + z^2$	0	0
(3) $x^2 + y^2 - z^2$	0	0
(4) $x^2 + y^2$	0 *and* $\mathrm{d}x \wedge \mathrm{d}y$	0 *or* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
(5) $x^2 - y^2$	0 *and* $\mathrm{d}x \wedge \mathrm{d}y$	0 *or* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
(6) x^2	0 *and* $\mathrm{d}x \wedge \mathrm{d}y$	$0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$ $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & -\alpha & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1-\alpha \end{pmatrix}$

(B) $k = (0, 0, 1)^T$, *i.e.*, $\hat{k} = \frac{\partial}{\partial z}$

Quadratic function f	*2-cocycles* ω	*Derivation D*
(7) 0	0	$0, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix},$ $\begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
(8) $a(x^2 + y^2)$	0	0 *and* $\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & 0 \end{pmatrix}$

Quadratic function f	*2-cocycles ω*	*Derivation D*
$(9)_1$ $a(x^2-y^2),\ a\neq\frac{1}{4}$	0	0 *and* $\begin{pmatrix} I_{2\times 2} & 0\\ 0 & 0\end{pmatrix}$
$(9)_2$ $\frac{1}{4}(x^2-y^2)$,	0 *and* $\mathrm{d}x\wedge\mathrm{d}z$	$0,\begin{pmatrix} I_{2\times 2} & 0\\ 0 & 0\end{pmatrix},\ \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$
(10) x^2	0	0 *and* $\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$

where $a>0,\ \alpha,\ \beta\in\mathbb{R}$ *are constants.*

Proof. Throughout the proof, 2-chain $\omega=\alpha\mathrm{d}y\wedge\mathrm{d}z+\beta\mathrm{d}z\wedge\mathrm{d}x+\gamma\mathrm{d}x\wedge\mathrm{d}y$ will be denoted by $\omega=(\alpha,\beta,\gamma)$.

(1) Assume that $\omega_1=(\alpha_1,\beta_1,\gamma_1)$, $\omega_2=(\alpha_2,\beta_2,\gamma_2)$, where $\alpha_1,\ \beta_1,\ \gamma_1$ are not zero at the same time and so are $\alpha_2,\ \beta_2,\ \gamma_2$, and A is a matrix of which the adjoint matrix $\widetilde{A}$ satisfies $\widetilde{A}\omega_2=\omega_1$, then $\overline{A}=\begin{pmatrix}A&0\\0&1\end{pmatrix}$ is the isomorphism from $\mathfrak{g}_{\omega_1}$ to $\mathfrak{g}_{\omega_2}$. In particular, we can choose $\omega=\mathrm{d}x\wedge\mathrm{d}y$ as the standard form.

(2) and (3) follows from Corollaries 4.2 and 4.5.

(4) and (5). By Corollary 4.2 and the fact that if $\omega_1-\omega_2$ is exact, $\mathfrak{g}_{\omega_1}$ is isomorphic to $\mathfrak{g}_{\omega_2}$, we only need to consider the case $\omega=\gamma\mathrm{d}x\wedge\mathrm{d}y$, where $\gamma\neq 0$. Let $\omega_1=\gamma_1\mathrm{d}x\wedge\mathrm{d}y$ and $\omega_2=\gamma_2\mathrm{d}x\wedge\mathrm{d}y$, $\gamma_1\neq 0,\ \gamma_2\neq 0$, $\mathfrak{g}_{\omega_1}$ is isomorphic to $\mathfrak{g}_{\omega_2}$ is obvious since $A=\begin{pmatrix}I_{3\times 3}&0\\0&\frac{\gamma_2}{\gamma_1}\end{pmatrix}$ is the isomorphism and we can choose $\omega=\mathrm{d}x\wedge\mathrm{d}y$ as the standard form. By Corollary 4.5 and the fact that if D_1-D_2 is exact, $\mathfrak{g}_{D_1}$ is isomorphic to $\mathfrak{g}_{D_2}$, we only need to consider the case $D=\begin{pmatrix}d\cdot I_{2\times 2}&0\\0&0\end{pmatrix}$, where $d\neq 0$. $\mathfrak{g}_{D_1}$ is isomorphic to $\mathfrak{g}_{D_2}$ is obvious since $D=\begin{pmatrix}I_{3\times 3}&0\\0&\frac{d_1}{d_2}\end{pmatrix}$ is the isomorphism. And we can choose $D=\begin{pmatrix}I_{2\times 2}&0\\0&0\end{pmatrix}$ as the standard form.

(6) As in (4) and (5), we only need to consider the case $\omega=(0,\beta,\gamma)$, assume that $\omega_1=(0,\beta_1,\gamma_1)$ and $\omega_2=(0,\beta_2,\gamma_2)$. Let $B\in GL(2)$ that satisfies $B\omega_2=\omega_1$, then $A=\begin{pmatrix}\sqrt[3]{\det(B)}&0&0\\0&\frac{1}{\sqrt[3]{\det(B)}}B&0\\0&0&1\end{pmatrix}$ is the isomorphism from $\mathfrak{g}_{\omega_1}$ to $\mathfrak{g}_{\omega_2}$. As for *D-extension*, we only need to consider the case $D=\begin{pmatrix}\mathrm{tr}(\overline{D})&0\\0&\overline{D}\end{pmatrix}$ by Corollary 4.5, and $\mathfrak{g}_{D_1}$ is isomorphic to $\mathfrak{g}_{D_2}$ if and only if $\overline{D_1}$ is similar to the matrix that is nonzero multiples of the matrix similar to $\overline{D_2}$. In fact, if $\overline{D_1}=d\overline{B}^{-1}\overline{D_2B}$ for some $\overline{B}\in GL(2)$, then $B=\begin{pmatrix}\det(\overline{B})&0&0\\0&\overline{B}&0\\0&0&d\end{pmatrix}$ is the isomorphism from $\mathfrak{g}_{D_1}$ to $\mathfrak{g}_{D_2}$.

The proof of the triviality of central extensions in the Cases (7), (8), (10) and in the Case $(9)_1$ are same because of Corollary 4.2. In the Case $(9)_2$, if the 2-cocycle ω is exact, from Corollary 4.2, we have $\alpha=-\beta$. So we only need to consider the case $\omega=\beta\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}$ and $\mathfrak{g}_{\omega_1}$ is isomorphic to $\mathfrak{g}_{\omega_2}$ is obvious since $B=\begin{pmatrix}I_{3\times 3}&0\\0&\frac{\beta_2}{\beta_1}\end{pmatrix}$ is the isomorphism, where $\omega_1=\beta_1\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}$, $\omega_2=\beta_2\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}$.

The proof of the determination of derivation D in Case (B) when $k=(0,0,1)^T$ listed in Theorem 3.3 is almost the same as the proof of the unimodular case and we leave it to the interest of the reader. ■

Now combine Proposition 2.3, Theorem 3.3 and 5.1, we obtain the following theorem that gives the classification of Lie–Poisson structures on $\mathbb{R}^4$.

Theorem 5.2. *Any* 4*-dimensional Lie algebra is isomorphic to one of the following forms which are considered as the extension by a derivation of some unimodular* 3*-dimensional Lie algebra listed in Theorem* 3.3(A).

Quadratic function f	*Derivation D*
(1) 0	$0, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} \delta & 1 & 0 \\ -1 & \delta & 0 \\ 0 & 0 & \epsilon \end{pmatrix}$
(2) $x^2 + y^2 + z^2$	0
(3) $x^2 + y^2 - z^2$	0
(4) $x^2 + y^2$	0 *and* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
(5) $x^2 - y^2$	0 *and* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
(6) x^2	$0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & -\varepsilon & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1-\zeta \end{pmatrix}$

where $\alpha, \beta, \gamma, \delta, \epsilon, \varepsilon, \zeta \in \mathbb{R}$ *are arbitrary constants.*

Proof. By Proposition 2.3, we know that any 4-dimensional Lie algebra is the extension of some 3-dimensional unimodular Lie algebra and Proposition 5.1 gives the classification of the extension of a fixed Lie algebra. The only thing left now is to decide the isomorphism of the extension of different Lie algebras, however this is easily done. This completes the proof. ■

By Proposition 5.1 and the statement at the beginning of this section, we classify all of the *affine Poisson structures* on $\mathbb{R}^3$.

Theorem 5.3. *On* $\mathbb{R}^3$*, any affine Poisson structure is isomorphic to one of the following forms:*
(A) $k = 0$ (*unimodular case*) (B) $k = (0, 0, 1)^T$*, i.e.,* $\hat{k} = \frac{\partial}{\partial z}$

Quadratic function f	*Affine Poisson structure*	*Quadratic function f*	*Affine Poisson structure*
(1) 0	$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$	(7) 0	$\pi_{k,f}$
(2)(3) $x^2 + y^2 \pm z^2$	π_f	(8) $a(x^2 + y^2)$	$\pi_{k,f}$
(4) $x^2 + y^2$	$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \pi_f$	$(9)_1$ $a(x^2 - y^2),\ a \neq \frac{1}{4}$	$\pi_{k,f}$
(5) $x^2 - y^2$	$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \pi_f$	$(9)_2$ $\frac{1}{4}(x^2 - y^2)$,	$\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + \pi_{k,f}$
(6) x^2	$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \pi_f$	(10) x^2	$\pi_{k,f}$

where $a > 0$ *is a constant.*

As an application of the classification of 4-dimensional Lie algebras, we give an example to describe conformal symplectic structure of corresponding linear Jacobi structure obtained by the decomposition of linear Poisson structures on $\mathbb{R}^4$. Pursuing this geometric approach of Jacobi structure is very interesting and we have another paper to study it. For more details about Jacobi structure, conformal symplectic structure and contact structure, please see [5,7].

Example 5.4. Consider the 4-dimensional Lie algebra decided by $f = x^2$ and the derivation $D = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$, the corresponding linear Poisson structure is

$$\pi = 2x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \frac{1}{2} x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4} + \frac{1}{4} x_2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} + \frac{1}{4} x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$$

and $k = D(\pi) = (0, 0, 0, 1)$, so the corresponding Jacobi structure is

$$(E, \Lambda) = \left(\frac{1}{3} \frac{\partial}{\partial x_4},\ 2x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \frac{1}{6} x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4} - \frac{1}{12} x_2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} - \frac{1}{12} x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \right).$$

After some straightforward computations we have, if $x_1 \neq 0$, the character distribution is 4-dimensional and the 2-form Ω, inverse to the bi-vector Λ, is

$$\Omega = \frac{1}{4} \frac{x_3}{x_1^2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \frac{1}{4} \frac{x_2}{x_1^2} \mathrm{d}x_3 \wedge \mathrm{d}x_1 + \frac{6}{x_1} \mathrm{d}x_4 \wedge \mathrm{d}x_1 + \frac{1}{2x_1} \mathrm{d}x_3 \wedge \mathrm{d}x_2$$

and $\omega = i_E \Omega = \frac{2}{x_1} \mathrm{d}x_1$. So we have

$$d\Omega = -\omega \wedge \Omega = \frac{1}{x_1^2} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3.$$

This shows that (Ω, ω) is the the conformal symplectic structure on the leaf.

If $x_1 = 0$, it is evident that the character distribution through $(0, x_2^0, x_3^0, x_4^0)$ is 2-dimensional if x_2^0, x_3^0 are not zero at the same time and the leaf is just the half-plane decided by x_4-axis and the point $(0, x_2^0, x_3^0, x_4^0)$ with x_4-axis omitted. The character distribution through $(0, 0, 0, x_4^0)$ is 1-dimensional and the leaf is just x_4-axis. ■

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